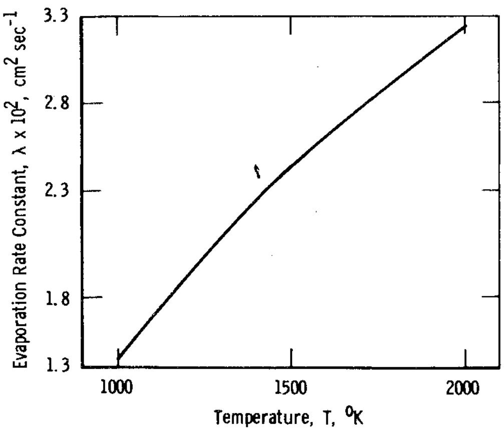

Fig. 2 Evaporation rate constant for jet-fuel vs mean reactor temperature (data from Ref. 1).

Using standard Laplace Transform tables, one obtains

$$(\alpha/\alpha^o) = y e^{-y}(d^2/dy^2)\{(\pi/-y)^{1/2}\operatorname{erf}(-y)^{1/2}\} = (e^{-y}/y)[\tfrac{3}{4}(\pi/-y)^{1/2}\operatorname{erf}(-y)^{1/2} - e^{+y}(\tfrac{3}{2}-y)]$$

From Ref. 6,

$$w(z) = e^{-z^2}\operatorname{erf}c(-iz) = e^{-z^2}[1-\operatorname{erf}(-iz)]$$

$$\operatorname{erf}(-y)^{1/2} = \mathscr{I}_m\{e^y[1-w(-y)^{1/2}]\} = i e^y \mathscr{I}_m[w(y)^{1/2}]$$

Therefore

$$\alpha/\alpha^o = (1/y)\{\tfrac{3}{4}(\pi/y)^{1/2}\mathscr{I}_m[w(y)^{1/2}] - (\tfrac{3}{2}-y)\}$$

Note: $\mathscr{I}_m[w(z)] = 2/(\pi)^{1/2}$ times Dawson's Integral.

Results and Discussion

Equation (6) was evaluated for a range of values of $d_o{}^2/\lambda\tau$ and the results are plotted in Fig. 1. Equation (3) was also evaluated and the results plotted in Fig. 1 to indicate the difference in fuel evaporation rate when recirculation is taken into account. It is hoped that use of Eq. (6) instead of Eq. (3) in stirred reactor calculations may improve pollutant predictions. Finally, values of λ, the evaporation rate constant, were evaluated and plotted in Fig. 2 as a function of reactor temperature T. Data for the convection correction to λ were obtained from Ref. 1.

Conclusions

An expression has been developed for the mean liquid fuel concentration in an intensely recirculating flow region. The expression allows liquid fuel evaporation rates to be evaluated with finite fuel concentrations predicted even for residence times greater than the critical value for single drops. The mean fuel evaporation rate may thus be significantly smaller than that for single drops. In a jet engine combustor in which strong recirculation is generated the fuel will not be fully evaporated.

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Analytical Approximate Calculation of Optimal Low-Thrust Energy Increase Trajectories

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Introduction

AN approximate analytic solution to optimal low thrust constant acceleration energy increase trajectories is presented. The spacecraft to be considered shall ascend from a circular orbit around the spherical central body to some specified energy level in minimal time. Analytical solutions to this problem have previously been found by Lawden,[1] whose solution does not predict the oscillatory character of the optimal control program. The solution by Breakwell and Rauch[2] holds an accuracy of about 1% over a small number of revolutions around the central body only. Jacobson and Powers[3] developed an accurate small parameter perturbation solution with a correct prediction of the oscillatory nature of the optimal control program. They all used the classical Newtonian formulation of the equations of motion.

The solution presented in this Note is based on a two-variable asymptotic expansion[4] of the equations of motion and the Lagrange multiplier equations. The regularized form[5] of the equations of motion leads to a very high accuracy of the first-order approximation over a large number of revolutions and allows to meet the boundary conditions very accurately.

Analysis

The spacecraft is treated as a point mass. It is equipped with a low thrust constant acceleration engine. All further perturbations are neglected. The regularized equations of motion for the planar ascent can be written as

$$r' = u \tag{1a}$$

$$u' = \mu + 2hr + \varepsilon r^2 e_r \tag{1b}$$

$$v' = \varepsilon r^2 e_\phi \tag{1c}$$

$$h' = \varepsilon(ue_r + ve_\phi) \tag{1d}$$

$$\phi' = v/r \tag{1e}$$

$$t' = r \tag{1f}$$

where r is the radius, ϕ the central angle, u and v are the components of the velocity in radial and circumferential direction, t is the time, and ε the constant small acceleration. The total energy h is used as an additional state variable. The components e_r and e_ϕ of the control $\vec{e}$ stand for

$$e_r = \cos\theta = \cot\theta(1+\cot^2\theta)^{-1/2} \tag{2a}$$

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$$e_\phi = \sin\theta = (1+\cot^2\theta)^{-1/2} \tag{2b}$$

Equation (1f) describes the time transformation law for the regularization. The prime denotes the derivative with respect to a new independent variable s, the regularized time.

The performance index Φ which has to be minimized is defined as

$$\Phi = \varepsilon t_f + v(h - h_f) \tag{3}$$

where v is an unknown constant Lagrange multiplier. Maximizing the Hamiltonian

$$H = \lambda_r u + \lambda_u(\mu + 2hr) + \lambda_\phi v/r + \lambda_t r + \varepsilon[\lambda_u r^2 e_r + \lambda_v r^2 e_\phi + \lambda_h(u e_r + v e_\phi)] \tag{4}$$

with respect to the control results in the optimal control function

$$\cot\theta = \lambda_u r^2 + \lambda_h u/(\lambda_v r^2 + \lambda_h v) \tag{5}$$

From H the Euler Lagrange equations are obtained as

$$\lambda_r' = -[\lambda_u(2h+\varepsilon r e_r) + \lambda_v \varepsilon 2 r e_\phi + \lambda_t] \tag{6a}$$
$$\lambda_u' = -(\lambda_r + \varepsilon\lambda_h e_r) \tag{6b}$$
$$\lambda_v' = -\lambda_h \varepsilon e_\phi \tag{6c}$$
$$\lambda_h' = -\lambda_u 2r \tag{6d}$$
$$\lambda_\phi' = 0 \tag{6e}$$
$$\lambda_t' = 0 \tag{6f}$$

The dimensionless boundary conditions with the initial state as a scaling system are determined as

$$\begin{gathered} r(0) = r_i = 1, \quad u(0) = u_i = 0, \quad v(0) = v_i = 1, \\ h(0) = h_i = -0.5, \quad \phi(0) = \phi_i = 0, \quad t(0) = t_i = 0, \\ h(s_f) = h_f, \quad \lambda_r(s_f) = \lambda_{rf} = 0, \quad \lambda_u(s_f) = \lambda_{uf} = 0, \\ \lambda_v(s_f) = \lambda_{vf} = 0, \quad \lambda_h(s_f) = \partial\Phi/\partial h(s_f) = v, \\ \lambda_\phi(s_f) = \lambda_{\phi f} = 0, \quad \lambda_t(s_f) = \partial\Phi/\partial t(s_f) = \varepsilon. \end{gathered} \tag{7}$$

An analytical approximate solution of this two-point boundary value problem shall be obtained by a two-variable expansion procedure, which requires the introduction of two new independent variables

$$\vartheta = \varepsilon s \quad \text{and} \quad \tau = \int_0^s \omega(\vartheta)\, d\sigma \tag{8}$$

The unknown function $\omega(\vartheta)$ has to be determined in the course of the problem solution. The state variables and the multipliers are expanded into asymptotic power series of the small parameter ε and are considered to be functions of v and τ.

$$x = X^{(0)} + \varepsilon X^{(1)} + \varepsilon^2 X^{(2)} + \dots \tag{9}$$

x stands for any state or multiplier variable, the superscript numbers denote the order of the expansion.

Note that a derivative with respect to s becomes

$$d(\cdot)/ds = \varepsilon \cdot \partial(\cdot)\ /\partial\vartheta + \omega \cdot \partial(\cdot)/\partial\tau \tag{10}$$

Substitution of the expansion terms and the new independent variables into Eqs. (1) and (6) and integration of the partial differential equations for each order finally leads to the approximate solution.

First-Order Solution

For the state variables the first order solution yields

$$r = R^{(0)} + \varepsilon(A\sin\tau + B\cos\tau) + 0(\varepsilon^2) \tag{11a}$$
$$u = \varepsilon[R^{(0)-1/2}(A\cos\tau - B\sin\tau) + 2R^{(0)5/2}] + 0(\varepsilon^2) \tag{11b}$$
$$v = R^{(0)1/2} + 0(\varepsilon) \tag{11c}$$
$$h = -(2R^{(0)})^{-1} + 0(\varepsilon) \tag{11d}$$

with $R^{(0)}$ as zero-order radius approximation $R^{(0)} = (1-3\vartheta)^{-2/3}$ Integration of the second term of Eq. (8) gives

$$\tau = (1 - R^{(0)-2})/4\varepsilon \tag{12}$$

Equations (11a–11d) contain the following abbreviations

$$A = C_1 R^{(0)5+(10)^{1/2}/4} + C_2 R^{(0)5-(10)^{1/2}/4} \qquad C_1 = -(2+C_2),$$

$$C_2 = \frac{R_f^{(0)7/4}\cos\tau_f + 2[3+(10)^{1/2}]R_f^{(0)(10)^{1/2}/4}}{[3-(10)^{1/2}]R_f^{(0)-(10)^{1/2}/4} - [3+(10)^{1/2}]R_f^{(0)(10)^{1/2}/4}}$$

$$B = C_3(R^{(0)5+(10)^{1/2}/4} - R^{(0)5-(10)^{1/2}/4})$$

$$C_3 = -\frac{R_f^{(0)7/4}\sin\tau_f}{[3+(10)^{1/2}]R_f^{(0)(10)^{1/2}/4} - [3-(10)^{1/2}]R_f^{(0)-(10)^{1/2}/4}}$$

The Euler-Lagrange Eqs. (6) may also be integrated to a first order solution

$$\lambda_r = \varepsilon(C\sin\tau + D\cos\tau) + 0(\varepsilon^2) \tag{13a}$$
$$\lambda_u = \varepsilon[R^{(0)1/2}\cdot(C\cos\tau - D\sin\tau) + R^{(0)} + 2\lambda_v^{(0)}R^{(0)2}] + 0(\varepsilon^2) \tag{13b}$$
$$\lambda_v = \lambda_v^{(0)} + \varepsilon L_v^{(1)} + 0(\varepsilon^2) \tag{13c}$$
$$\lambda_h = \lambda_h^{(0)} + \varepsilon[-2R^{(0)3/2}\cdot(-C\cos\tau + D\sin\tau) + L_h^{(1)}] + 0(\varepsilon^2) \tag{13d}$$

with the zero order values for

$$\lambda_v^{(0)} = -1/3\cdot\ (R^{(0)-1} - R^{(0)1/2}R_f^{(0)-3/2})$$
$$\lambda_h^{(0)} = -1/3\cdot\ (2R^{(0)1/2} + R^{(0)2}R_f^{(0)-3/2})$$

and the abbreviations

$$C = \frac{1}{648}\times R^{(0)-5/12}\exp\left\{\frac{1}{18}\left[1-\left(\frac{R^{(0)}}{R_f^{(0)}}\right)^{3/2}\right]\right\}\times [17(C_1I_1 + C_2I_2) + (C_1I_3 + C_2I_4)\times R_f^{(0)-3/2} + C_4]$$

$$D = \frac{1}{648}\times R^{(0)-5/12}\times\exp\left\{\frac{1}{18}\left[1-\left(\frac{R^{(0)}}{R_f^{(0)}}\right)^{3/2}\right]\right\}\times [17C_3(I_1 - I_2) + C_3(I_3 - I_4)R_f^{(0)-3/2} + C_5]$$

$$C_4 = -648\times R_f^{(0)11/12}\times\cos\tau_f + 17(C_1I_{1f} + C_2I_{2f}) + (C_1I_{3f} + C_2I_{4f})\times R_f^{(0)-3/2}$$

$$C_5 = 648\times R_f^{(0)11/12}\times\sin\tau_f + 17C_3(I_{1f} - I_{2f}) + C_3(I_{3f} - I_{4f})\times R_f^{(0)-3/2}$$

$$I_{1,2} = \frac{48}{-7\pm 3(10)^{1/2}}R^{(0)-7\pm 3(10)^{1/2}/12} + \frac{12R_f^{(0)9/4}}{2\pm 3(10)^{1/2}}\times R^{(0)2\pm 3(10)^{1/2}/12} + \frac{72}{-10\pm 3(10)^{1/2}}R^{(0)-10\pm 3(10)^{1/2}/12} - \frac{24R_f^{(0)-3/2}}{8\pm 3(10)^{1/2}}R^{(0)8\pm 3(10)^{1/2}/12}$$

$$I_{3,4} = \frac{48}{11\pm 3(10)^{1/2}}R^{(0)11\pm 3(10)^{1/2}/12} + \frac{12R_f^{(0)9/4}}{20\pm 3(10)^{1/2}}\times R^{(0)20\pm 3(10)^{1/2}/12} + \frac{72}{8\pm 3(10)^{1/2}}R^{(0)8\pm 3(10)^{1/2}/12} - \frac{24R_f^{(0)-3/2}}{26\pm 3(10)^{1/2}}R^{(0)26\pm 3(10)^{1/2}/12}$$

The terms $L_v^{(1)}$ and $L_h^{(1)}$ are obtained from the solution of the differential equations

$$\partial L_v^{(1)}/\partial\vartheta = -L_h^{(1)}$$

and

$$\partial^2 L_v^{(1)}/\partial\vartheta^2 - 4R^{(0)3}L_v^{(1)} = -R^{(0)1/2}(AD - BC)$$

The optimal control program results in

$$\cot\theta = -R^{(0)}\lambda_u^{(1)} - R^{(0)-1}\lambda_h^{(0)}u^{(1)} \tag{14}$$

Remember that for energy increase trajectories the only specified state variable at t_f was h_f which lead to the unknown boundary condition $\lambda_{hf} = v$. This means that the analytic approximate solution depends on the multiplier v. An approximate determination of v was found by an expansion of H to first order at t_f. As $H_f = 0$ for an autonomous system with unspecified t_f the approximate value of v is found as $v = \lambda_{hf}^{(0)} = -R_f^{(0)1/2}$. This result is already substituted in the first-order solution described above.

Time and Central Angle Approximation

The analytical first-order solution does not include the histories of t and ϕ. Fortunately, approximations for t and ϕ may be found by an independent procedure. The expansion of the time Eq. (1f) leads to an approximation with poor convergence characteristics. Hence it is advantageous to replace t by

$$t = t^* + \Delta t \qquad \text{with} \qquad t^* = \int_0^s R^{(0)}\, d\sigma = (1 - R^{(0)-1/2})/\varepsilon \qquad (15)$$

For Δt an expansion solution may be found

$$\Delta t = -\varepsilon[R^{(0)1/2}(A\cos\tau - B\sin\tau) - \tfrac{9}{7}\times(1 - R^{(0)}) + 2] \qquad (16)$$

The two-variable expansion of the ϕ-equation (1e) gives a first-order approximation

$$\phi = \tau + \varepsilon R^{(0)-1}[(A\cos\tau - B\sin\tau) + \tfrac{9}{4}\times(R^{(0)2} - 1) + 2] \qquad (17)$$

Both results Eqs. (16) and (17) include the assumption that the second-order expansion term of the energy, $H^{(2)}$, and that part of $v^{(2)}$ which depends on ϑ only, $V^{(2)}$, be zero, i.e. $H^{(2)} = V^{(2)} = 0$.

Discussion

The asymptotic solution shows the basic characteristics of the spiral trajectory. The zero-order solution $v^{(0)} = R^{(0)1/2}$ is equivalent to the relation $rv^2 = 1$, which usually is regarded as the standard circular asymptotic solution of spiral type trajectories.[6] Furthermore, the zero-order radius approximation is equivalent to the well-known spiral distance approximation $r^{(0)}(t) = (1 - \varepsilon t)^{-2}$.

The derived solution has a singularity at $\vartheta = \frac{1}{3}$. This value corresponds to the escape condition $h = 0$ and leads to $r(\vartheta = \frac{1}{3}) \to \infty$. This is the well-known behavior of the Kepler solution of the equations of motion without thrust. Consequently, the solution is valid only for energy levels less than zero.

Near escape thrust acceleration and gravity force are of the same order of magnitude, and the solution ceases to possess two different time scales. This change in the physical characteristics violates the basic assumption for the procedure which implies the dominance of the gravity force. In addition the expansion solution assumes that the radial velocity component u is small compared to v. This condition does not hold near escape, too.

Examples

To check the accuracy, the analytic solution is compared with the results of Jacobson and Powers[3] and with numerical calculations. 1) Jacobson and Powers used the specifications $h_f = -0.2904134$ with $\varepsilon = 1.189409 \times 10^{-3}$. These lead to a scaled minimum time $t_f = 200$. The number of revolutions of the optimal trajectory is slightly more than 22. The coincidence of these results with the two-variable expansion solution lies between 3 and 6 significant digits for the different state variables and multipliers over the whole region $0 \leq t \leq t_f$. Figures 1 and 2 show the control angle and the radial distance histories as functions of the nondimensional time. One sees immediately

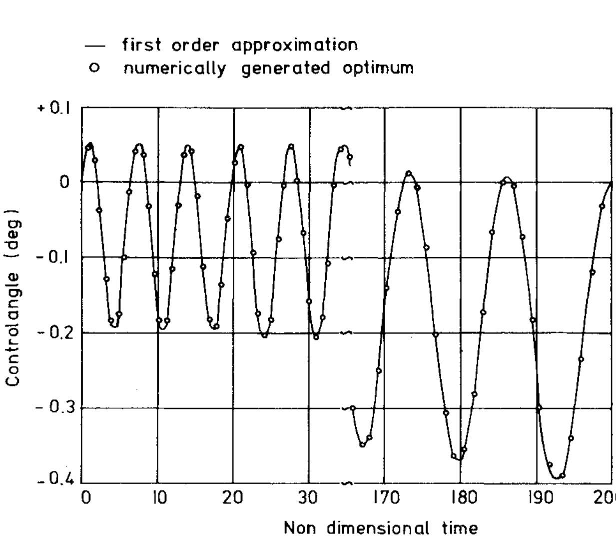

Fig. 1 Time history of the control angle (measured against the path tangent).

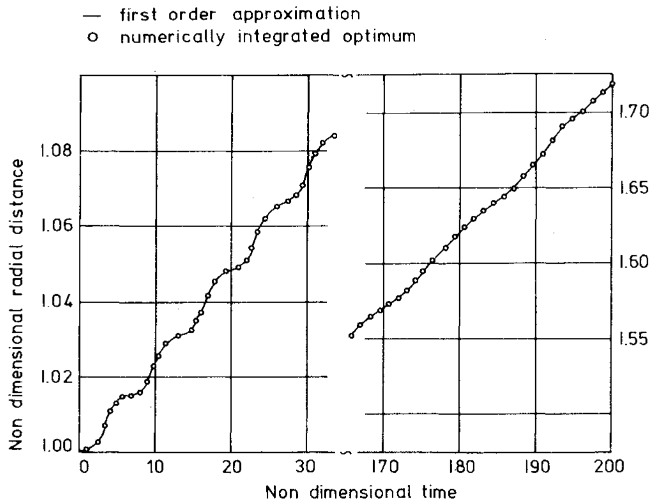

Fig. 2 Time history of the radial distance.

how well the approximation matches the changing amplitude and period of the trajectory and control angle oscillations.

2) The numerical procedure was calculated with $h_f = -0.195$ and $\varepsilon = 1.1242371 \times 10^{-4}$. This results in a scaled $t_f = 1\,885.5$ and more than 300 revolutions. The analytical approximation and the numerically integrated trajectory coincide within at least 3 significant digits over the whole region. A numerical optimization program was started with Lagrange multipliers at $t_i = 0$ which resulted from the approximate solution. This extremely sensitive optimization process converged within few iterations. Time and centriangle approximations are accurate within 5 digits for both examples.

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Thermal Buckling of Orthotropic Cylindrical Shells

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Introduction

THERMAL buckling plays an important role in the design of thin walled structures subjected to thermal environment. The existing studies on thermal buckling are confined to homogeneous, isotropic and ring stiffened shells.[1–7] Dasgupta and

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